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HEAT TRANSFER FROM A PLATE IN A COMPRESSIBLE GAS FLOW

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Аннотация—Из решения сопряженной задачи теплообмена показано, что температура поверхности раздела пластина-жидкость является неоналитической функцией расстояния вдоль пластины, имеющей точки ветвления при $x = 0$ и ∞. Отсюда вытекает невозможность априорного задания температуры поверхности, а также непригодность обычного определения коэффициента теплообмена (последнее было отмечено ранее в фундаментальной работе [1]).

NOMENCLATURE

- x, longitudinal coordinate along the plate;
- y, transverse coordinate;
- ρ, density of gas;
- μ, dynamic viscosity of gas;
- u, longitudinal component of gas velocity;
- v, transverse component of gas velocity;
- Θ, gas temperature;
- T, temperature of the body;
- k_f, gas thermal conductivity;
- k_s, thermal conductivity of the plate;
- $r(0)$, recovery factor;
- c_p, specific heat at constant pressure.

Subscripts

- ω, on the plate surface;
- ∞, in the bulk of the flow.

1. STATEMENT AND SOLUTION OF THE PROBLEM

CONSIDER aerodynamic heating of a thin plate by a gas flow (Fig. 1). The system of equations for a laminar compressible boundary layer is written as follows

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{1}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{2}$$

$$\rho u \frac{\partial \Theta}{\partial x} + \rho v \frac{\partial \Theta}{\partial y} = \frac{1}{Pr}\cdot\frac{\partial}{\partial y}\left(\mu \frac{\partial \Theta}{\partial y}\right) + \frac{\mu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

(assuming that $Pr = \text{const.}$, $C_p = \text{const.}$) at ordinary boundary conditions

$$u|_{y=0} = 0, \qquad u|_{y=\infty} = U_\infty, \tag{4}$$

$$\Theta|_{y=0} = \Theta_\omega(x), \qquad \Theta|_{y=\infty} = T_\infty. \tag{5}$$

As shown by Chapman and Rubesin [1], a good approximation of the temperature dependence of the viscosity is

$$\frac{\mu}{\mu_\infty} = C\frac{\Theta}{T_\infty}, \tag{6}$$

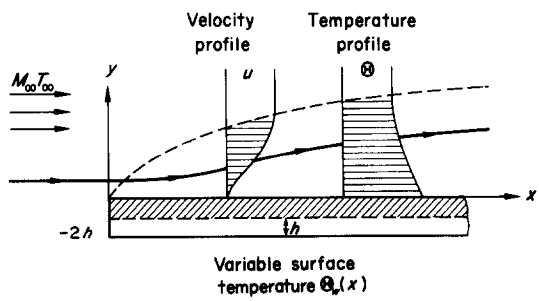

FIG. 1.

where

$$C = \sqrt{\left(\frac{\bar{\Theta}_\omega}{T_\infty}\right)\frac{T_\infty + S}{\Theta_\omega + S}} \qquad (7)$$

($\bar{\Theta}_\omega$ is the mean surface temperature; S a constant).

Expression for the plate temperature is as follows

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q(x, y)}{k_s}, \qquad (8)$$

for example, with conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0. \qquad (9)$$

For a symmetrical flow past a plate $2h$ thick and at $Q(x, -y) = Q(x, y)$

$$\left.\frac{\partial T}{\partial y}\right|_{y=-h} = 0. \qquad (10)$$

Later the function $Q(x, y)$ is assumed continuous along x and y and analytic along x.

At the interface usual conditions hold:

$$\left.\left(-k_f(\Theta)\frac{\partial \Theta}{\partial y}\right)\right|_{y=0} = -k_s\left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad (11)$$

and

$$\Theta(x, 0) \equiv \Theta_\omega(x) = \Theta_l + \tau(x) = T|_{y=0}, \qquad (12)$$

where Θ_l is the temperature of a heat-insulated surface.

The problem is either to determine the temperature of the plate and heat flux through it or to find the kind of a heat source $Q(x, y)$ necessary for the temperature of the surface and heat flux through it to assume the prescribed values.

Equation (8) for a thin plate may be written with the use of conditions (10)–(12) in the form

$$\frac{d^2\Theta_\omega(x)}{dx^2} - \frac{1}{k_s h}q(x) = -\frac{W(x)}{k_s}, \qquad (13)$$

where $q(x)$ is a heat flux through the interface plate-flow and $W(x)$ is an averaged source. As was shown above $W(x) = \sum_{n=0}^{\infty} W_n x^n$.

Conditions (9) transform to

$$\left.\frac{d\Theta_\omega(x)}{dx}\right|_{x=0} = \left.\frac{d\Theta_\omega(x)}{dx}\right|_{x=L} = 0. \qquad (14)$$

Energy equation (3) may be written in the form

$$\frac{\partial^2 \Theta^*}{\partial \eta^2} + Pr\,.\,f\frac{\partial \Theta^*}{\partial \eta} - 2Pr\,f'x^*\frac{\partial \Theta^*}{\partial x^*} = -\frac{Pr}{4}(\gamma - 1)M_\infty^2(f'')^2 \qquad (15)$$

with dimensionless variables

$$x^* = \frac{x}{L}, \qquad \Theta^* = \frac{\Theta}{T_\infty}. \qquad (16)$$

The new variable η is determined by the relationship

$$f(\eta) = \frac{\psi^*}{(\sqrt{x^*})}, \qquad (17)$$

where ψ^* is a dimensionless stream function

$$\psi^* = \frac{\psi}{\sqrt{(\nu_\infty U_\infty L C)}} \qquad (18)$$

and $f(\eta)$ – the solution of the Blasius equation

$$f''' + ff'' = 0, \qquad (19)$$

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 2.$$

The boundary conditions for equation (15) will be written in the form

$$\Theta^*(x^*, 0) \equiv \Theta_\omega^*(x^*) = \Theta_l^* + \tau^*(x^*),$$
$$\Theta^*(x^*, \infty) = 1. \qquad (20)$$

The partial solution of linear inhomogeneous equation (15) may be found in the form

$$\Theta^*(x^*, \eta) = N(\eta) = 1 + \frac{\gamma - 1}{2} . M_\infty^2 . r(\eta), \quad (21)$$

where

$$r(\eta) = \frac{Pr}{2} \int_\eta^\infty [f''(\xi)]^{Pr} \int_0^\xi [f''(\zeta)]^{2-Pr} \, d\zeta \, d\xi. \quad (22)$$

Thus the thermal equilibrium temperature is

$$\Theta_t = N(0) = T_\infty \left[1 + r(0) \frac{\gamma - 1}{2} M_\infty^2\right], \quad (23)$$

where $r(0)$ is a recovery factor

$$r(0) = \frac{Pr}{2} \int_0^\infty [f''(\xi)]^{Pr} d\xi \left(\int_0^\xi [f''(\eta)]^{2-Pr} d\eta \right). \quad (24)$$

Solution (21) satisfies the boundary conditions

$$\begin{aligned} &\Theta^*(x^*, 0) = \Theta_t^*, \\ &\Theta^*(x^*, \infty) = 1, \\ &\frac{\partial \Theta^*}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0. \end{aligned} \quad (25)$$

The solution of the homogeneous equation

$$\frac{\partial^2 \Theta^*}{\partial \eta^2} + Pr . f \frac{\partial \Theta^*}{\partial \eta} - 2Pr \, f' x^* \frac{\partial \Theta^*}{\partial x^*} = 0 \quad (26)$$

may be found by the method of separation of variables. Assuming $\Theta^* = X(x^*)Y(\eta)$, we obtain

$$\frac{1}{f'Y}(Y'' + Pr . f Y') = 2Pr \, x^* . \frac{X'}{X} = \text{const.} \quad (27)$$

Assume the separation constant equal to $2Pr . p$, where $p = \alpha n, \alpha n + \beta, \alpha n + \gamma$ $(n = 0, 1, 2, \ldots)$. Choice of numbers $\alpha, \beta, \gamma > 0$ will be shown later.

From equation (27) we have

$$X_p(x^*) = x^{*p}. \quad (28)$$

The functions $Y_p(\eta)$ are determined from the equation

$$Y_p'' + Pr f Y_p' - 2Pr f' p Y_p = 0 \quad (29)$$

and should satisfy the boundary conditions

$$Y_p(0) = 1, \qquad Y_p(\infty) = 0 \quad (30)$$

(whether such solutions exist will be established later).

Equation (26) is linear. Therefore summation of the solutions of the form

$$\begin{aligned} &\bar{\bar{a}}_n x^{*\alpha n} \quad . \, Y_{\alpha n}(\eta), \\ &p_n x^{*\alpha n + \beta} . \, Y_{\alpha n + \beta}(\eta), \\ &q_n x^{*\alpha n + \gamma} . \, Y_{\alpha n + \gamma}(\eta) \end{aligned} \quad (31)$$

(where $\bar{\bar{a}}_n$, p_n, q_n are some constants) yield the solution to equation (26)

$$\begin{aligned} \Theta^*(x^*, \eta) = \sum_{n=0}^\infty [\bar{\bar{a}}_n Y_{\alpha n}(\eta) + p_n Y_{\alpha n + \beta}(\eta) x^{*\beta} \\ + q_n Y_{\alpha n + \gamma}(\eta) \, x^{*\gamma}] \, x^{*\alpha n}, \end{aligned} \quad (32)$$

satisfying the boundary conditions

$$\left.\begin{aligned} &\Theta^*(x^*, 0) = \sum_{n=0}^\infty [\bar{\bar{a}}_n + p_n x^{*\beta} + q_n x^{*\gamma}] \, x^{*\alpha n}, \\ &\Theta^*(x^*, \infty) = 0. \end{aligned}\right\} \quad (33)$$

Thus, the complete solution to the energy equation is

$$\begin{aligned} \Theta^*(x^*, \eta) = N(\eta) + \sum_{n=0}^\infty [\bar{\bar{a}}_n Y_{\alpha n}(\eta) \\ + p_n Y_{\alpha n + \beta}(\eta) x^{*\beta} + q_n Y_{\alpha n + \gamma}(\eta) x^{*\gamma}] x^{*\alpha n} \end{aligned} \quad (34)$$

and it satisfies the boundary conditions

$$\begin{aligned} &\Theta_\omega^* \equiv \Theta^*(x^*, 0) = \sum_{n=0}^\infty [a_n + p_n x^{*\beta} + \\ &\qquad + q_n x^{*\gamma}] \, x^{*\alpha n}, \\ &\Theta^*(x^*, \infty) = 1, \end{aligned} \quad (35)$$

where

$$a_0 \overset{df}{=} \Theta_t^* + \bar{\bar{a}}_0, \qquad a_n \overset{df}{=} \bar{\bar{a}}_n \qquad \text{at} \quad n = 1, 2, \ldots$$

The determination of the coefficients a_n, p_n, q_n is conducted using equation (13). First the expression for the heat flux in terms of dimensionless variables is written down. Since

$$q(x) = -\left[k_f(\Theta)\frac{\partial \Theta}{\partial y}\right]_{y=o}$$

$$= -\frac{k_\infty T_\infty}{2}\cdot C_\omega \sqrt{\left(\frac{u_\infty}{\nu_\infty x C}\right)}\cdot\sum_{n=0}^{\infty}[\bar{\bar{a}}_n Y'_{\alpha n}(0)$$

$$+ p_n Y'_{\alpha n+\beta}(0)x^{*\beta} + q_n Y'_{\alpha n+\gamma}(0)\, x^{*\gamma}]\, x^{*\alpha n}, \quad (36)$$

where

$$C_\omega(x) = \sqrt{\left[\frac{\Theta_\omega(x)}{T_\infty}\right]}\frac{T_\infty + S}{\Theta_\omega(x) + S}, \quad (37)$$

then, in terms of dimensionless variables of (16), and with

$$G_w(x^*) = [\Theta_w^*(x^*)]\cdot\frac{1 + G}{\Theta_w^*(x^*) + G}, \quad (38)$$

where

$$G = \frac{S}{T_\infty},$$

one gets

$$q(x^*) = -B_1\sqrt{\left[\frac{\Theta_\omega^*(x^*)}{x^*}\right]}\cdot\frac{\sum_{n=0}^{\infty}[\bar{\bar{a}}_n Y'_{\alpha n}(0) + p_n Y'_{\alpha n+\beta}(0)\, x^{*\beta} + q_n Y'_{\alpha n+\gamma}(0)\, x^{*\gamma}]\, x^{*\alpha n}}{\Theta_\omega^*(x^*) + G}, \quad (39)$$

where

$$B_1 = \frac{k_\infty T_\infty(1 + G)}{2}\cdot\sqrt{\left(\frac{u_\infty}{\nu_\infty L C}\right)}. \quad (40)$$

Finally equation (13) in dimensionless coordinates takes the form

$$\frac{d^2\Theta_\omega^*}{dx^{*2}} + B\sqrt{\left[\frac{\Theta_\omega^*(x^*)}{x^*}\right]}\cdot\frac{\sum_{n=0}^{\infty}[\bar{\bar{a}}_n Y'_{\alpha n}(0) + p_n Y'_{\alpha n+\beta}(0)\, x^{*\beta} + q_n Y'_{\alpha n+\gamma}(0)\, x^{*\gamma}]\, x^{*\alpha n}}{\Theta_\omega^*(x^*) + G} = V(x^*), \quad (41)$$

where

$$B = \frac{B_1 L^2}{k_s T_\infty h}, \qquad V(x^*) = -\frac{W(Lx^*)\,.\,L^2}{k_s T_\infty}$$

$$= \sum_{n=0}^{\infty} V_n x^{*n}, \qquad V_n = -\frac{L^{2+n}}{k_s T_\infty}\,.\,W_n. \quad (42)$$

Thus according to (34) and (35) the temperature of the plate surface, i.e. the solution to equation (41) is sought in the form

$$\Theta_\omega^*(x^*) = \sum_{n=0}^{\infty}(a_n + p_n x^{*\beta} + q_n x^{*\gamma})\, x^{*\alpha n}. \quad (43)$$

When substituting (43) into (41), it is noted that in order to obtain the identity it is necessary that the numbers αn, $\alpha n + \beta$, $\alpha n + \gamma$ form a commutative additive semi-group including all non-negative integers. Therefore $\alpha = \frac{3}{2}$ is assumed. Then $\beta = \frac{1}{2}$, $\gamma = 1$. Hence equation (43) may be presented in the form

$$\Theta_\omega^* = \sum_{n=0}^{\infty}(a_n + p_n x^{*\frac{1}{2}} + q_n x^*)\, x^{*\frac{3}{2}n}. \quad (44)$$

Let S_n be the coefficients of some power series with fractional powers of x obtained by multiplying the power series of the same type by the coefficients M_n and R_n. Denote

$$S_n = \{M_n R_n\}, \quad (45)$$

where

$$S_n = \sum_{\kappa=0}^{n} M_\kappa R_{n-\kappa}, \ (n = 0, 1, 2, \ldots).$$

Then (41) yields the recurrent relationships

$$\left.\begin{aligned}
&\tfrac{3}{2}n(\tfrac{3}{2}n-1)a_n + B[\{\bar{a}_{n-1}A_{n-1}\} + \{\bar{p}_{n-2}Q_{n-2}\} + \{\bar{q}_{n-2}P_{n-2}\}] \\
&\qquad = \begin{cases} V_{[\frac{3}{2}n-2]} & \text{if } [\tfrac{3}{2}n-2] = \tfrac{3}{2}n-2; \\ 0 & \text{if } [\tfrac{3}{2}n-2] \neq \tfrac{3}{2}n-2. \end{cases} \\
&(\tfrac{3}{2}n+\tfrac{1}{2})(\tfrac{3}{2}n-\tfrac{1}{2})p_n + B[\{\bar{p}_{n-1}A_{n-1}\} + \{\tilde{a}_{n-1}P_{n-1}\} + \{\bar{q}_{n-2}Q_{n-2}\}] \\
&\qquad = \begin{cases} V_{[\frac{3}{2}n-\frac{3}{2}]} & \text{if } [\tfrac{3}{2}n-\tfrac{3}{2}] = \tfrac{3}{2}n-\tfrac{3}{2}; \\ 0 & \text{if } [\tfrac{3}{2}n-\tfrac{3}{2}] \neq \tfrac{3}{2}n-\tfrac{3}{2}. \end{cases} \\
&\tfrac{3}{2}n(\tfrac{3}{2}n+1)q_n + B[\{\bar{p}_{n-1}P_{n-1}\} + \{\bar{q}_{n-1}A_{n-1}\} + \{\tilde{a}_{n-1}Q_{n-1}\}] \\
&\qquad = \begin{cases} V_{[\frac{3}{2}n-1]} & \text{if } [\tfrac{3}{2}n-1] = \tfrac{3}{2}n-1; \\ 0 & \text{if } [\tfrac{3}{2}n-1] \neq \tfrac{3}{2}n-1, \end{cases}
\end{aligned}\right\} \tag{46}$$

where $\bar{a}_n$, $\bar{p}_n$, $\bar{q}_n$ are determined from the recurrent relationships

$$\left.\begin{aligned}
a_n &= c_{n,1} + 2\{\bar{p}_{n-1}\bar{q}_{n-1}\}, \\
p_n &= c_{n-1,3} + 2\{\bar{a}_n\bar{p}_n\}, \\
q_n &= c_{n,2} + 2\{\bar{a}_n\bar{q}_n\}, \\
&(n = 0, 1, 2, \ldots),
\end{aligned}\right\} \tag{47}$$

$$c_{m,j} = \frac{1}{m\bar{a}_{0,j}} \cdot \sum_{k=1}^{m} (3k - m)\bar{a}_{k,j}c_{m-k,j}, \tag{48}$$

$$(m = 1, 2, \ldots; j = 1, 2, 3),$$

where $c_{0,j} = \bar{a}_{0,j}^2$; $\bar{a}_{k,1} = \bar{a}_k$; $\bar{a}_{k,2} = \bar{p}_k$; $\bar{a}_{k,3} = \bar{q}_k$, $k = 0, 1, 2, \ldots$, and A_n, P_n and Q_n are determined from the recurrent relationships

$$\begin{aligned}
\bar{\bar{a}}_n Y'_{\frac{3}{2}n}(0) &= \{\tilde{a}_nA_n\} + \{p_{n-1}Q_{n-1}\} \\
&\quad + \{q_{n-1}P_{n-1}\}, \\
p_nY'_{\frac{3}{2}n+\frac{1}{2}}(0) &= \{p_nA_n\} + \{\tilde{a}_nP_n\} \\
&\quad + \{q_{n-1}Q_{n-1}\}, \qquad (49) \\
q_nY'_{\frac{3}{2}n+1}(0) &= \{p_nP_n\} + \{q_nA_n\} \\
&\quad + \{\tilde{a}_nQ_n\},
\end{aligned}$$

where $\tilde{a}_0 = a_0 + G$, $\tilde{a}_k = a_k$, $k \geqslant 1$.

In all the recurrent relationships the summands containing negative subscripts are considered equal to zero.

The analysis of equations (47) shows that the coefficients $\Theta^*_\infty(x^*)$:

$$a_1, a_2, \ldots; p_0, p_1, \ldots; q_1, q_2, \ldots$$

are sought in terms of a_0, q_0, of coefficients of source V_n and of values

$$Y'_{\frac{3}{2}n}(0), \qquad Y'_{\frac{3}{2}n+\frac{1}{2}}(0), \qquad Y'_{\frac{3}{2}n+1}(0)$$
$$(n = 0, 1, 2, \ldots).$$

Conditions (14) give two equations for determination of a_0 and q_0 (from the recurrent equations (47) $p_0 = 0$):

$$\left.\begin{aligned}
\left.\frac{d\Theta^*_\infty}{dx^*}\right|_{x^*=0} &= q_0 = 0, \\
\left.\frac{d\Theta^*_\infty}{dx^*}\right|_{x^*=1} &= \sum_{n=1}^{\infty} [\tfrac{3}{2}na_n + (\tfrac{3}{2}n + \tfrac{1}{2})p_n \\
&\qquad + (\tfrac{3}{2}n + 1)q_n] = 0.
\end{aligned}\right\} \tag{50}$$

Let us write out the first twelve coefficients of the function Θ_∞^* (here $Y_p'(0) = \gamma_p$)

$$a_1 = -\frac{4B\bar{\bar{a}}_0\gamma_0\sqrt{a_0}}{3\tilde{a}_0},$$

$$a_2 = \frac{V_1}{6} + \frac{B^2\bar{\bar{a}}_0\gamma_0[\bar{\bar{a}}_0\gamma_0(\tilde{a}_0 - 2a_0) + 2a_0\tilde{a}_0\gamma_{\frac{1}{2}}]}{9\tilde{a}_0^3},$$

$$a_3 = -\frac{B}{126\tilde{a}_0^3 a_0^{\frac{3}{2}}} \{4\tilde{a}_0 a_0[2a_0\tilde{a}_0\gamma_3 + \gamma_0\bar{\bar{a}}_0(\tilde{a}_0 - 2a_0)]\, a_2 + [4\tilde{a}_0^2 a_0\gamma_{\frac{1}{2}} - 4a_0\tilde{a}_0\bar{\bar{a}}_0\gamma_0 - \tilde{a}_0^2\bar{\bar{a}}_0\gamma_0 - 8a_0^2(\tilde{a}_0\gamma_{\frac{1}{2}} - \bar{\bar{a}}_0\gamma_0)]\, a_1^2\},$$

$$p_0 = 0,$$

$$p_1 = \frac{V_0}{2}, \tag{51}$$

$$p_2 = -\frac{BV_0}{35\tilde{a}_0^2 a_0^{\frac{1}{2}}}[\bar{\bar{a}}_0\gamma_0(\tilde{a}_0 - 2a_0) + 2a_0\gamma_2],$$

$$p_3 = \frac{V_3}{20} - \frac{1}{12\tilde{a}_0^4 a_0^{\frac{3}{2}n}} [\![\{3\tilde{a}_0^2[2\tilde{a}_0 a_0(\gamma_{\frac{1}{2}} + \gamma_2) - \bar{\bar{a}}_0\gamma_0(4a_0 + \tilde{a}_0)]\, a_1 + 16\bar{\bar{a}}_0 a_0^{\frac{3}{2}} B\gamma_0(\gamma_{\frac{1}{2}}\tilde{a}_0 - 2\bar{\bar{a}}_0\gamma_0\gamma_2\tilde{a}_0)\}\, p_1 + 6a_0\tilde{a}_0^2\,[\tilde{a}_0^2 + 2a_0(\tilde{a}_0\gamma_{\frac{1}{2}} - \bar{\bar{a}}_0\gamma_0)]\, p_2]\!],$$

$$q_0 = q_1 = 0,$$

$$q_2 = \frac{V_2}{12},$$

$$q_3 = \frac{B}{198\tilde{a}_0^3 a_0^{\frac{3}{2}}}\{[4a_0\tilde{a}_0(\bar{\bar{a}}_0\gamma_0 - \tilde{a}_0\gamma_2) + \bar{\bar{a}}_0\tilde{a}_0^2\gamma_0 + 8a_0^2(\gamma_2\tilde{a}_0 - \gamma_0\bar{\bar{a}}_0)]\, p_1^2 + 4a_0\tilde{a}_0[2a_0(\bar{\bar{a}}_0\gamma_0 - \tilde{a}_0\gamma_4) - \tilde{a}_0\bar{\bar{a}}_0\gamma_0]\, q_2\}.$$

If at large x the source is described by whole negative powers of x (i.e. $V = \sum_{n=0}^{-\infty} V_n x^n$), then $\Theta^*(x^*, \eta)$ will be sought in the form of (32) where $n = 0, -1, -2, \ldots$. Then in equations (33)–(43) it should also be considered that $n = 0, -1, -2, \ldots$. It is seen that at rather large x the solution for Θ_∞^* is formally found in the form

$$\Theta_\infty^* = \sum_{n=0}^{-\infty} (a_n + p_n x^{*\frac{1}{2}} + q_n x^*)\, x^{*\frac{3}{2}n}. \tag{52}$$

Here the recurrent relationships remain valid and the note that the summands with negative subscripts should be equated to zero no longer applies. It should be remembered here that Y_p are already different, since p attains the values $\frac{3}{2}n$, $\frac{3}{2}n + \frac{1}{2}$ and $\frac{3}{2}n + 1$, where $n = 0, -1, -2, \ldots$.

Equations (51) remain valid if the signs of all the subscripts are reversed. In equation (48) the sum is taken from $k = -1$ to m where $m = -1, -2, \ldots$.

2. DEMONSTRATION OF THE CONVERGENCE OF SERIES (44)

It is important to show that series (44) is asymptotic and moreover converges at small $|x|$. For this purpose Wiener's assumption is used that if the function $x(\varphi)$ can be expanded into an absolutely convergent Fourier series and does not become equal to zero, then $1/x(\varphi)$ can also be expanded into an absolutely convergent Fourier series. His Tauberian theorem [2–4] and fractional power of closed operators will also be used.

Let X represent a B-space and $\{T_t;\ t \geqslant 0\} \subseteq L(X, X)$ a continuous semi-group of class (C_0) of equal powers. Let us further introduce the function

$$f_{t,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} e^{z\lambda - tz^\alpha}\, dz & \text{at } \lambda \geqslant 0, \\ 0 & \text{at } \lambda < 0, \end{cases} \tag{53}$$

where $a > 0$, $t > 0$, $0 < \alpha < 1$ and also the branch z^α chosen so that $Re(z^\alpha) > 0$ for $Re(z) > 0$.

This branch is a single-valued function in a complex z-plane with a cut along the negative section of the real axis. Following Bochner [5] it may be shown that the operators determined by equations

$$T_{t,\alpha}x \equiv \hat{T}_t x = \begin{cases} \int_0^\infty f_{t,\alpha}(s) T_s x \, ds & \text{at } t > 0, \\ x & \text{at } t = 0 \end{cases} \quad (54)$$

represent continuous groups of class (C_0) of equal powers and the operator family $\{\hat{T}_t\}$ forms a holomorphic semi-group. It appears here that the infinitesimal producing operator $\hat{A} = \hat{A}_\alpha$ of the semi-group $\{T_t\}$ is related to the infinitesimal operator of the semi-group $\{T_t\}$ by the equation

$$\hat{A}_\alpha x = -(-A)^\alpha x \qquad \text{for all} \quad x \in D(A),\dagger$$

where the fractional powers $(-A)^\alpha$ of the operator $(-A)$ are determined by the equality

$$(-A)^\alpha x = \Gamma(-\alpha)^{-1} \int_0^\infty \lambda^{-\alpha-1} (T_\lambda - I)\, x \, d\lambda, \qquad x \in D(A), \quad (55)$$

and the form of the resolvant of the operator $\hat{A}_\alpha$ was obtained by Kato [6].

Further if substitution

$$z = \frac{xy' - y}{x^2},$$

is made into (41) it is not difficult to show that series (44) converges at small $|x|$, since it becomes clear that in this case the Hukuhara theorem holds [7] on the existence of a fixed point in the functional space.

It is now clear that the temperature of the plate surface is not an analytic function of whole powers of x in the vicinity of the point (0.0).

3. HEAT TRANSFER COEFFICIENT

In the case of variable surface temperature the heat-transfer coefficient is determined by the formula

$$\alpha = \frac{q}{\Theta_\omega(x) - \Theta_l}. \quad (56)$$

For the calculation of α, the earlier obtained expression $q(x^*)$:

$$q(x^*) = -\frac{k_\infty T_\infty}{2} C_\omega(x^*) \sqrt{\left(\frac{u_\infty}{v_\infty L x^* C}\right)} \cdot \sum_{n=0}^{\infty} [\bar{\bar{a}}_n Y'_{\frac{1}{2}n}(0) + p_n x^{*\frac{1}{2}} Y'_{\frac{1}{2}n+\frac{1}{2}}(0) + q_n x^* Y'_{\frac{1}{2}n+1}(0)]\, x^{*\frac{1}{2}n}.$$

is compared with the equality

$$q(x^*) = \alpha T_\infty [\Theta^*_\omega(x^*) - \Theta^*_l].$$

Taking into account that

$$\bar{\bar{a}}_0 = \Theta^*(0,0) - \Theta^*_l,$$

$$p_0 = q_0 = 0,$$

gives

$$\alpha = \bar{\alpha} . \frac{C_\omega(x^*)[\Theta^*(0,0) - \Theta^*_l]}{C[\Theta^*_\omega(x^*) - \Theta^*_l]} - \frac{k_\infty C_\omega(x^*) \sqrt{\left(\frac{u_\infty}{v_\infty L x^* C}\right)} \cdot \sum_{n=1}^{\infty} [\bar{\bar{a}}_n Y'_{\frac{1}{2}n}(0) + p_n x^{*\frac{1}{2}} Y'_{\frac{1}{2}n+\frac{1}{2}}(0) + q_n x^* Y'_{\frac{1}{2}n+1}(0)]\, x^{*\frac{1}{2}n}}{2[\Theta^*_\omega(x^*) - \Theta^*_l]}, \quad (57)$$

where $\bar{\alpha}$ is heat-transfer coefficient at $\Theta_\omega \equiv$ const.

† $D(A)$ is the range of the infinitesimal producting operator of the semigroup $\{T_t\}$.

4. CALCULATION OF EIGENFUNCTIONS $Y_p(\eta)$

According to [1] the functions of temperature distribution $Y_n(\eta)$ are found by integration of equations

$$Y_n'' + Pr\, f\, Y_n' - 2Pr\, nf'\, Y_n = 0 \qquad (58)$$

at boundary conditions

$$Y_n(0) = 1, \qquad Y_n(\infty) = 0, \qquad (59)$$

where $n = 0, 1, 2, \ldots$ and $f(\eta)$ and $f'(\eta)$ satisfy the Blasius equation (19).

In [1] it has been noted that for large η, $f(\eta)$ approaches a linear function and $f'(\eta)$ approaches a constant. For $\eta > 4{\cdot}1$ within the accuracy of four decimal places

$$f(\eta) = 2(\eta - 0{\cdot}86038), \qquad (60)$$
$$f'(\eta) = 2.$$

It has also been noted in [1] that while solving equation (58) cumbersome calculations may be avoided if its asymptotic solution is found. However, the general asymptotic solution given is not good enough for the boundary-value problem, (58) and (59).

Consider equation (29)

$$Y_p'' + Pr\, f\, Y_p' - 2Pr\, pf'\, Y_p = 0, \qquad (61)$$

where p is any real number

$$f(\eta) = a(\eta - b),$$

$$f'(\eta) = a, \qquad (62)$$

where $a > 0$ and b are any real numbers and the boundary conditions are of the form

$$Y_p(0) = 1, \qquad Y_p(\infty) = 0. \qquad (63)$$

Under these conditions the existence of the only solution satisfying the boundary conditions (63) may be demonstrated (when $p \geqslant 0$). Substitution

$$x = (\sqrt{Pr})(\eta - b) \qquad (64)$$

leads to

$$Y_p'' + axY_p' - 2paY_p = 0. \qquad (65)$$

By applying successive substitutions to (61)

$$Y_p = \exp\left(-\frac{ax^2}{4}\right) z_p, \qquad V = \frac{z_p'}{z_p}, \qquad (66)$$

the equation

$$V' + V^2 - \left(\frac{a^2x^2}{4} + \frac{a}{2} + 2ap\right) = 0 \qquad (67)$$

is obtained.

Finally, substitution

$$V = \frac{ax}{2} + G(x) \qquad (68)$$

gives

$$G' = 2ap - axG - G^2. \qquad (69)$$

It may be shown that there exists a family of solutions to (69) which may be represented by an asymptotic series

$$G(x) \simeq h_0 + h_1 x^{-1} + h_2 x^{-2} + \ldots \qquad (x \to \infty). \qquad (70)$$

To be more correct, such $x^\circ > 0$ and $N > 0$ can be found that for $G^0 \in [-N, N]$ the solution with the initial condition $G(x^0) = G$ may be infinitely continued to the right and is represented in the form of (70). Here $h_0, h_1, \ldots,$ do not depend on G^0. These solutions are asymptotically steady according to Lyapunov.

For (69) series (70) will be a series of odd powers ($h_{2m} = 0$, $m = 0, 1, 2, \ldots$). The recurrent relationship for the coefficients of the series is of the form

$$ah_{2k+1} = (2k - 1)\, h_{2k-1} - \sum_{\substack{m+p=k-1 \\ m,p=0,1,2\ldots}} h_{2m+1} h_{2p+1}, \qquad (71)$$

where $k = 1, 2, \ldots$ and $h_1 = 2p$.

Thus, for example,

$$h_3 = \frac{1}{a} 2p(1 - 2p),$$

$$h_5 = \frac{1}{a^2} 2p(1 - 2p)(3 - 4p).$$

So, for (65) there exists a family of solutions represented by the asymptotic series

$$Y_p \simeq cx^{2p}(1 + r_1 x^{-2} + r_2 x^{-4} + \ldots),\ x \to \infty, \quad (72)$$

where formally

$$1 + r_1 t + r_2 t^2 + \ldots \overset{df}{=} \exp\left(-\frac{h_3}{2}t - \frac{h_5}{4}t^2 - \ldots\right). \quad (73)$$

Now, on application of substitution

$$V = -\frac{ax}{2} + G(x) \quad (74)$$

to (71)

$$G' = a(1 + 2p) + axG + G^2. \quad (75)$$

It may be shown that equation (75) possesses *only one* solution $G(x)$ which is represented by the asymptotic series

$$G(x) \simeq l_0 + l_1 x^{-1} + l_2 x^{-2} + \ldots (x \to \infty), \quad (76)$$

and this solution is unsteady and is the *only one* within the range $x^0 < x < \infty$.

When applied to equation (65) this means that within the accuracy of $C = \text{const.} \neq 0$ the only solution exists of the form

$$Y_p \simeq cx^{-(1+2p)} \exp\left(-\frac{a^2x^2}{2}\right) .(1 + s_1 x^{-2} + s_2 x^{-4} + \ldots),\quad x \to \infty, \quad (77)$$

where formally

$$1 + s_1 t + s_2 t^2 + \ldots \overset{df}{=} \exp\left(-\frac{l_3}{2}t - \frac{l_5}{4}t^2 - \ldots\right). \quad (78)$$

Series (76) contains only odd powers of x($l_{2m} = 0$, $m = 0, 1, 2, \ldots$). $l_1 = -(1 + 2p)$ and coefficients l_{2k+1}, $k = 1, 2, \ldots$ are determined from the recurrent relationship

$$al_{2k+1} = -(2k - 1)\,l_{2k-1} - \sum_{\substack{m+p=k-1\\ m,p=0,1,2\ldots}} l_{2m+1}\,l_{2p+1}. \quad (79)$$

For example,

$$l_3 = -\frac{2p(1 + 2p)}{a},$$

$$l_5 = \frac{-2p(1 + 2p)(1 - 4p)}{a^2}.$$

Since it is evident that any solution to Y_{p_1} from the family (70) and solution to Y_{p_2} of the form (77) are linearly independent then the general solution of (65) is of the form*

$$Y_p = C_1 Y_{p_1} + C_2 Y_{p_2}.$$

Hence it is clear that the fulfilment of the boundary condition (63) yields $C_1 = 0$ (at $p \geqslant 0$). Therefore the asymptotic solution of equation (65) is presented in the form

$$Y_p = x^{-(1+2p)} \exp\left(-\frac{a^2x^2}{2}\right) (1 + s_1 x^{-2} + s_2 x^{-4} + \ldots),\quad x \to \infty. \quad (80)$$

Y'_p being found from the above equation, numerical integration may be started to determine $Y_p(0)$.

5. DISCUSSION

In conclusion let us emphasize an important finding from the solution of the conjugated problem (1)–(12). In work [1] by Chapman and Rubesin the surface temperature is set in the form of Taylor series in terms of powers x. The solution to the conjugated problem shows that this is incorrect because the surface temperature is not an analytic function of x, except in some trivial cases, but has its particular point at $x = 0$. Hence it follows that the surface temperature cannot be prescribed if it is variable.

* The general asymptotic solution of equation (1) is obtained for any large x and any real p.

The heat-transfer coefficient is determined by equation (57) in which the coefficients a_n, p_n, q_n and, therefore, the surface temperature $\Theta_\omega(x)$ are found from the above solution of the conjugated problem.

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Abstract—It is shown by the solution of a conjugated heat transfer problem that the temperature of the interface plate–liquid is a non-analytic function of the distance along the plate with the branch points at $x = 0$ and ∞. Hence it follows that an *a priori* assumption of the interface temperature is impossible and the ordinary determination of heat transfer, which has been pointed out earlier in [1], is inapplicable.

TRANSPORT DE CHALEUR À PARTIR D'UNE PLAQUE DANS UN ÉCOULEMENT GAZEUX COMPRESSIBLE

Résumé—On montre, grâce à la solution d'un problème de transport de chaleur conjugué, que la température de l'interface plaque–liquide est une fonction non-analytique de la distance le long de la plaque avec des points de branchement à $x = 0$ et ∞. Il s'ensuit donc qu'une hypothèse *a priori* sur la température de l'interface est impossible et que la détermination habituelle du transport de chaleur, qui a été signalé auparavant dans [1], est inapplicable.

WÄRMEÜBERGANG VON EINER PLATTE IN EINER KOMPRESSIBLEN GASSTRÖMUNG

Zusammenfassung—Auf Grund der Lösung eines konjugierten Wärmeübergangsproblems wird gezeigt, dass die Temperatur der Zwischenschicht zwischen Platte und Flüssigkeit eine nicht-analytische Funktion der Entfernung auf der Platte ist, mit Verzweigungspunkten bei $x = 0$ und ∞. Daraus folgt, dass eine a priori Annahme für die Zwischenschichttemperatur unmöglich ist und die übliche Berechnung des Wärmeübergangs, wie sie in [1] angedeutet ist, nicht durchgeführt werden kann.